Spherically Symmetric, Metrically Static, Isolated Systems in Quasi-Metric Gravity

by

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Abstract

Working within the quasi-metric framework (QMF) described elsewhere, we examine the gravitational field exterior respectively interior to a spherically symmetric, isolated body made of perfect fluid. By construction the system is "metrically static", meaning that its associated gravitational field is static except for the effects of the global cosmic expansion on the spatial geometry. To ensure that the global cosmic expansion will not induce instabilities in the fluid source and thus violating the metrically static condition, the equation of state of the fluid is required to take a particular form (fulfilled for, e.g., an ideal gas).

We set up dynamical equations for the gravitational field and give an exact solution for the exterior part. Furthermore we find equations of motion applying to inertial test particles moving in the exterior gravitational field. The metrically static condition implies that the radius of the source increases and that distances between circular orbits increase according to the Hubble law, but such that circle orbit velocities are unaffected. This means that the dynamically measured mass of the source increases linearly with cosmic scale, or that the gravitational "constant" increases in time. We show that, if this model of an expanding gravitational field is taken to represent the gravitational field of the solar system, this has no serious consequences for observational aspects of planetary motion. On the contrary some observational facts of the Earth-Moon system are naturally explained within the QMF.

1 Introduction

The idea that the cosmic expansion may possibly be relevant for local systems came up many years ago; see, e.g., [1] and references therein. More recently there has been renewed interest in this idea; in part because it has become clear that there is no compelling observational evidence showing the expected deviations from the global Hubble law for

galaxies in the vicinity of or even within the local group of galaxies. See, e.g., [2] and references therein.

Even if there are in principle no direct observations ruling out the relevance of the global Hubble law on local scales, the generally accepted view is that for all practical purposes, local systems may be treated as decoupled from the cosmic expansion. This view reflects predictions coming from the standard framework of metric gravity. That is, it is well-known that metric theory predicts that realistic local systems are hardly affected at all by the cosmological expansion (its effect should at best be totally negligible, see, e.g., [1] and references therein). The reason for this prediction is basically that in metric theory, the cosmological expansion must be modelled within a mathematical framework where space-time is postulated to be a semi-Riemannian manifold. However, when analyzing the influence of the cosmological expansion on local systems there should be no reason to expect that predictions made within the metric framework should continue to hold in a theory where the structure of space-time is non-metric.

Recently a review of a new type of non-metric space-time framework, the so-called quasi-metric framework (QMF), was presented in [3]. Also presented was an alternative relativistic theory of gravity formulated within this framework. (A more detailed presentation can be found in [4].) This theory correctly predicts the "classic" solar system tests in the case where an asymptotically Minkowski background is invoked as an approximation and the cosmological expansion can be neglected. However, for reasons explained in [3], the theory is based on a $\mathbf{S}^3\times\mathbf{R}$-background rather than a Minkowski background as the global basic ("prior") geometry of the Universe. According to quasi-metric theory the nature of the cosmological expansion is a direct consequence of the existence of this $\mathbf{S}^3\times\mathbf{R}$-background and the expansion applies to all systems where gravitational dynamics dominates (hereafter called "gravitational systems"), regardless of scale. That is, in quasi-metric gravity the mathematical modelling of the Hubble expansion and thus its physical interpretation are different than in metric theory, and as a consequence, the Hubble expansion is predicted to influence local, gravitationally bound systems sufficiently that its effects should be observable in experiments. On the other hand, since quantum-mechanical states should be unaffected by the expansion, quasi-metric theory allows that the global cosmic expansion does not apply to quantum-mechanical systems bound by non-gravitational forces where gravitational interactions are negligible (hereafter called "atomic systems") [5].

To find more exactly how the cosmological expansion affects local gravitational systems according to the quasi-metric theory, one must first calculate the spherically symmetric gravitational field with the $\mathbf{S}^3\times\mathbf{R}$-background, both interior and exterior to the

source. Then one should use the quasi-metric equations of motion to calculate how test particles move in the exterior gravitational field. We show in section 3.3 of this paper that the quasi-metric theory predicts that the exterior gravitational field should expand according to the Hubble law. This also applies to the interior gravitational field if potential instabilities induced by the global cosmic expansion can be neglected (see section 3.2). In particular, for a source made of ideal gas the cosmic expansion induces no instabilities so the radius of a body made of ideal gas is predicted to expand. This result may support an interpretation of geological data indicating that the Earth is expanding according to the Hubble law, see reference [6] and references cited therein. (It is difficult to measure such a small expansion rate directly due to the existence of larger local displacements of the Earth's surface.) Besides, an expanding Earth should cause changes in its spin rate; we show in section 4.2 of this paper that the secular spin-down of the Earth as inferred from historical astronomical observations may in fact be of cosmological origin and only about half of the currently accepted value. Quasi-metric theory also predicts a cosmological origin of and different values for the recession of the Moon and its mean acceleration, other than those inferred from lunar laser ranging (LLR) experiments using standard theory. However, these differences are due to model-dependency since the LLR data yields that the recession of the Moon follows Hubble's law when analyzed within the QMF, and the quasi-metric predictions are consistent with a modern lunar ephemeris.

Finally it is shown that the predicted cosmic expansion of the solar system's gravitational field does not lead to easily detected perturbations in the observed motion of the planets. However, some less easily detected effects should be measurable; in fact a newly discovered secular increase of the astronomical unit may be explained by cosmic expansion. Also active mass is predicted to show a secular increase proportional to the Hubble parameter; in section 4.3 we argue that the predicted value of the increase is not in conflict with current test experiments. These results are very different from their counterparts in metric theory; that is why it is generally believed that observations confirm that the solar system is decoupled from the cosmic expansion when it is in fact the other way around.

2 Quasi-metric relativity in brief

2.1 General formulae

In this section we summarize the main features of the QMF and a quasi-metric theory of gravity. A considerably more extensive discussion can be found in [3] or [4].

The mathematical foundation of the QMF can be described by first considering a 5-dimensional product manifold $\mathcal{M}\times\mathbf{R}_1$, where $\mathcal{M}=\mathcal{S}\times\mathbf{R}_2$ is a (globally hyperbolic) Lorentzian space-time manifold, $\mathbf{R}_1$ and $\mathbf{R}_2$ are two copies of the real line and $\mathcal{S}$ is a compact Riemannian 3-dimensional manifold (without boundaries). Then *the global time function* t representing the extra (degenerate) time dimension $\mathbf{R}_1$ is introduced as a coordinate on $\mathbf{R}_1$. Moreover, for t given it is convenient to use a coordinate system $\{x^\mu\}$ (μ taking values in the interval $0-3$) where the ordinary time coordinate x^0 on $\mathcal{M}$ scales like ct; this ensures that x^0 is in some sense a mirror of t and thus a "preferred" global time coordinate. A coordinate system with a global time coordinate of this type we call a *global time coordinate system* (GTCS). Hence, expressed in a GTCS $\{x^\mu\}$, x^0 is interpreted as a global coordinate on $\mathbf{R}_2$ and $\{x^j\}$ (j taking values in the interval $1-3$) as spatial coordinates on $\mathcal{S}$. The class of GTCSs is a set of preferred coordinate systems inasmuch as the equations of quasi-metric relativity take special forms when expressed in a GTCS. Note that there exist infinitely many GTCSs.

The 4-dimensional quasi-metric space-time manifold $\mathcal{N}$ can now be defined by slicing the sub-manifold $x^0=ct$ (using a GTCS) out of the initial 5-dimensional space-time manifold. Furthermore, $\mathcal{N}$ is equipped with two families of Lorentzian space-time metric tensor fields $\bar{\mathbf{g}}_t$ and $\mathbf{g}_t$. The metric family $\bar{\mathbf{g}}_t$ represents a solution of field equations, and from $\bar{\mathbf{g}}_t$ one can construct the "physical" metric family $\mathbf{g}_t$ which is used when comparing predictions to experiments. It is convenient to think of the metric families as single degenerate metrics on (a subset of) $\mathcal{M}\times\mathbf{R}_1$, where the degeneracy manifests itself via the conditions $\bar{\mathbf{g}}_t(\frac{\partial}{\partial t},\cdot)\equiv0$, $\mathbf{g}_t(\frac{\partial}{\partial t},\cdot)\equiv0$. Finally, notice that $\mathcal{N}$ differs from a Lorentzian manifold and that this becomes evident only when it is equipped with an affine connection (see below).

From the above description we see that within the QMF, the canonical description of space-time is taken as fundamental. That is, quasi-metric space-time is constructed as consisting of two mutually orthogonal foliations: on the one hand space-time can be sliced up globally into a family of 3-dimensional space-like hypersurfaces (called the fundamental hypersurfaces (FHSs)) by the global time function t, on the other hand space-time can be foliated into a family of time-like curves everywhere orthogonal to the FHSs. These curves represent the world lines of a family of hypothetical observers called the fundamental observers (FOs). There exists a unique relationship between t and the proper time as measured by any FO.

Now one characteristic property of quasi-metric theory is that it postulates the existence of systematic scale changes between gravitational and atomic systems (and the main role of t is to describe the global aspects of such changes). This means that gravi-

tational quantities are postulated to exhibit an extra variation when measured in atomic units (and vice versa). One may think of this as if fixed operationally defined atomic units vary formally in space-time. Moreover, since c and Planck's constant $\hbar$ by definition are not formally variable, the formal variation of time units is equal to that of length units and inverse to that of mass units. We now postulate that this formal variation of atomic length (or time) units is defined from the geometry of the FHSs in $(\mathcal{N}, \bar{\mathbf{g}}_t)$. This means that the form of $\bar{\mathbf{g}}_t$ must be restricted such that all variation in the spatial geometry follows from a spatial scale factor $\bar{F}_t$. Thus by definition, measured in atomic units, the formal variability of gravitational quantities with the dimension of time or length goes as $\bar{F}_t$, whereas the the formal variability of gravitational quantities with the dimension of mass goes as $\bar{F}_t^{-1}$.

To determine the form of $\bar{F}_t$ we require that no extra arbitrary scale or parameter should be introduced (i.e., no characteristic scale should be associated with $\bar{F}_t$). This yields the (rather unique) choice $\bar{F}_t \equiv c\bar{N}_t t$, where $\bar{N}_t$ is the lapse function field family of the FOs in $(\mathcal{N}, \bar{\mathbf{g}}_t)$. With $\bar{F}_t$ given, together with the requirement that the FHSs should be compact and have a trivial topology, it is now straightforward to set up the general form of $\bar{\mathbf{g}}_t$. It thus can be argued [3, 4] that in a GTCS, the most general form allowed for the family $\bar{\mathbf{g}}_t$ may be represented by the family of line elements (we use the metric signature $(-+++)$ and Einstein's summation convention throughout)

$$\overline{ds}_t^2 = \bar{N}_t^2 \Big[(\bar{N}^i \bar{N}^j S_{ij} - 1)(dx^0)^2 + 2\frac{t}{t_0} \bar{N}^i S_{ij} dx^j dx^0 + \frac{t^2}{t_0^2} S_{ij} dx^i dx^j \Big], \tag{1}$$

where t_0 is an arbitrary reference epoch and where $S_{ij}dx^i dx^j$ denotes the metric of the 3-sphere $\mathbf{S}^3$ (with a radius equal to ct_0). Moreover, $\frac{t_0}{t}\bar{N}^k \frac{\partial}{\partial x^k}$ is the family of shift vector fields of the FOs in $(\mathcal{N}, \bar{\mathbf{g}}_t)$. (We may also define the family of shift covector fields $\frac{t}{t_0}\bar{N}_i \equiv \frac{t}{t_0}\bar{N}_t^2 S_{ik}\bar{N}^k$.) Note that, whereas $\bar{N}_t$ may depend on t, equation (1) does not contain any further dependence on t than what is shown explicitly. Also note that equation (1) may be taken as a postulate.

The time evolution of the spatial geometry of the FHSs is given by the change of $\bar{F}_t \equiv c\bar{N}_t t$ in the hypersurface-orthogonal direction. This evolution may be split up according to the definition (where a comma denotes partial derivation, the symbol '$\bar{\perp}$' denotes a scalar product with the unit normal vector field $-\bar{\mathbf{n}}_t$ of the FHSs, and where $\pounds_{\bar{\mathbf{n}}_t}$ denotes Lie derivation in the direction normal to the FHSs holding t constant)

$$\bar{F}_t^{-1} \overset{\star}{\pounds}_{\bar{\mathbf{n}}_t} \bar{F}_t \equiv \bar{F}_t^{-1} \Big((c\bar{N}_t)^{-1} \bar{F}_{t,t} + \pounds_{\bar{\mathbf{n}}_t} \bar{F}_t \Big) = \frac{1}{c\bar{N}_t t} + \frac{\bar{N}_{t,t}}{c\bar{N}_t^2} - \frac{\bar{N}_{t,\bar{\perp}}}{\bar{N}_t} \equiv c^{-2}\bar{x}_t + c^{-1}\bar{H}_t, \tag{2}$$

where $c^{-2}\bar{x}_t$ represents the *kinematical contribution* to the evolution of the spatial scale

factor and $c^{-1}\bar{H}_t$ represents the so-called *non-kinematical contribution* defined by

$$\bar{H}_t = \frac{1}{\bar{N}_t t} + \bar{y}_t, \qquad \bar{y}_t \equiv c^{-1}\sqrt{\bar{a}_{\mathcal{F}k}\bar{a}^k_{\mathcal{F}}}, \qquad c^{-2}\bar{a}_{\mathcal{F}j} \equiv \frac{\bar{N}_{t,j}}{\bar{N}_t}. \tag{3}$$

We see from equation (3) that the non-kinematical evolution (NKE) of the spatial geometry takes the form of an "expansion". Furthermore the NKE consists of two terms; the first term $\frac{1}{\bar{N}_t t}$ represents the *global NKE* due to the global curvature of the FHSs, whereas the second term $\bar{y}_t$ represents the *local NKE* coming from the gravitational field. This second term is not "realized" globally since it is absent in equation (2). Besides we see from equation (2) that the evolution of $\bar{N}_t$ with time may also be written as a sum of one kinematical and one non-kinematical term, i.e.

$$\frac{\bar{N}_{t,t}}{c\bar{N}_t^2} - \frac{\bar{N}_{t,\bar{\perp}}}{\bar{N}_t} = c^{-2}\bar{x}_t + c^{-1}\bar{y}_t. \tag{4}$$

The split-ups defined in equations (2), (3) and (4) are necessary to be able to construct $\mathbf{g}_t$ from $\bar{\mathbf{g}}_t$ [3]. Note that the kinematical evolution (KE) of the spatial scale factor may be positive or negative.

Next, on the quasi-metric space-time manifold $\mathcal{N}$ two linear, symmetric "degenerate" connections $\bar{\overset{\star}{\nabla}}$ and $\overset{\star}{\nabla}$ are defined. These connections are called degenerate due to the fact that they are essentially connections compatible with the 5-dimensional degenerate metrics $\bar{\mathbf{g}}_t$ and $\mathbf{g}_t$, respectively, on $\mathcal{M}\times\mathbf{R}_1$ and then just restricted to $\mathcal{N}$. In the following we describe the connection $\overset{\star}{\nabla}$ since this connection yields the quasi-metric equations of motion in $(\mathcal{N}, \mathbf{g}_t)$. That is, we introduce a torsion-free, metric-compatible 5-dimensional connection $\overset{\star}{\nabla}$ with the property that

$$\overset{\star}{\nabla}_{\frac{\partial}{\partial t}}\mathbf{g}_t = 0, \qquad \overset{\star}{\nabla}_{\frac{\partial}{\partial t}}\mathbf{n}_t = 0, \qquad \overset{\star}{\nabla}_{\frac{\partial}{\partial t}}\mathbf{h}_t = 0, \tag{5}$$

on $\mathcal{M}\times\mathbf{R}_1$ and consider the restriction of $\overset{\star}{\nabla}$ to $\mathcal{N}$. Here $\mathbf{h}_t$ is the spatial metric family intrinsic to FHSs and $\mathbf{n}_t$ is the unit vector family normal to the FHSs in $(\mathcal{N}, \mathbf{g}_t)$. It can be shown [4] that, expressed in a GTCS where the spatial coordinates do not depend on t, the components which do not vanish identically of the degenerate connection field are given by

$$\overset{\star}{\Gamma}{}^{\alpha}_{\mu t} \equiv \overset{\star}{\Gamma}{}^{\alpha}_{t\mu} \equiv \frac{1}{t}\delta^{\alpha}_{i}\delta^{i}_{\mu}, \qquad \overset{\star}{\Gamma}{}^{\alpha}_{\nu\mu} \equiv \frac{1}{2}g^{\alpha\sigma}_{(t)}\Big(g_{(t)\sigma\mu},_{\nu} + g_{(t)\nu\sigma},_{\mu} - g_{(t)\nu\mu},_{\sigma}\Big) \equiv \Gamma^{\alpha}_{(t)\nu\mu}. \tag{6}$$

The general equations of motion for test particles are identical to the geodesic equation obtained from $\overset{\star}{\nabla}$. In a GTCS they take the form (see [4] for a derivation)

$$\frac{d^2x^{\mu}}{d\lambda^2} + \Big(\overset{\star}{\Gamma}{}^{\mu}_{t\nu}\frac{dt}{d\lambda} + \Gamma^{\mu}_{(t)\beta\nu}\frac{dx^{\beta}}{d\lambda}\Big)\frac{dx^{\nu}}{d\lambda} = \Big(\frac{cd\tau_t}{d\lambda}\Big)^2 c^{-2}a^{\mu}_{(t)}, \tag{7}$$

where $d\tau_t$ is the proper time as measured along the curve, λ is some general affine parameter and $\mathbf{a}_t$ is the 4-acceleration as measured along the curve. From equations (6) and (7) we see that quasi-metric theory cannot be identified with any metric theory since the affine connection compatible with a general metric family is non-metric.

As mentioned above, a basic property of the QMF is that gravitational quantities will be formally variable when measured in atomic units. In particular this applies to the gravitational "constant" G. However, for convenience we *define* G to be constant so that its formal variability is transferred to the mass (and charge, if any). Thus, we have to separate between *active mass*, which is a scalar field, and *passive mass* (passive gravitational mass and inertial mass). By dimensional analysis it is found [4] that active mass m_t varies formally as $\bar{F}_t$ measured in atomic units (but passive mass does of course not vary). We then get

$$m_{t,t} = \Big(\frac{1}{t} + \frac{\bar{N}_{t,t}}{\bar{N}_t}\Big) m_t, \qquad m_{t,\bar{\perp}} = \frac{\bar{N}_{t,\bar{\perp}}}{\bar{N}_t} m_t, \qquad m_{t,j} = c^{-2} \bar{a}_{\mathcal{F}j} m_t, \tag{8}$$

where $\bar{\mathbf{a}}_{\mathcal{F}}$ is the 4-acceleration of the FOs in the family $\bar{\mathbf{g}}_t$. Taking into the account this variation of active mass in quasi-metric space-time it is possible to find local conservation laws. These local conservation laws involve the degenerate covariant divergence $\overset{\star}{\bar{\nabla}} \cdot \mathbf{T}_t$ of the active stress-energy tensor $\mathbf{T}_t$, and they take the form [4]

$$T^{\nu}_{(t)\mu \bar{*} \nu} = 2 \frac{\bar{N}_{t,\nu}}{\bar{N}_t} T^{\nu}_{(t)\mu} - \frac{2}{c \bar{N}_t} \Big(\frac{1}{t} + \frac{\bar{N}_{t,t}}{\bar{N}_t}\Big) T_{(t)\bar{\perp}\mu}, \tag{9}$$

where the symbol '$\bar{*}$' denotes degenerate covariant derivation $\overset{\star}{\bar{\nabla}}$ in component notation. Notice that these local conservation laws imply that inertial observers move along geodesics of $\overset{\star}{\bar{\nabla}}$ in $(\mathcal{N}, \bar{\mathbf{g}}_t)$, and that this guarantees that inertial observers move along geodesics of of $\overset{\star}{\nabla}$ in $(\mathcal{N}, \mathbf{g}_t)$ as well [3, 4]. This means that the equations of motion (7) are consistent with the local conservation laws (9).

It is useful to project these local conservation laws with respect to the FHSs. We then get the equations (in content equivalent to equation (9))

$$\overset{\star}{\mathcal{L}}_{\bar{\mathbf{n}}_t} T_{(t)\bar{\perp}\bar{\perp}} = \Big(\bar{K}_t - 2\frac{\bar{N}_{t,\bar{\perp}}}{\bar{N}_t} - \frac{2}{c\bar{N}_t}\Big[\frac{1}{t} + \frac{\bar{N}_{t,t}}{\bar{N}_t}\Big]\Big) T_{(t)\bar{\perp}\bar{\perp}} + \bar{K}_{(t)ik} \hat{T}^{ik}_{(t)} - \hat{T}^{i}_{(t)\bar{\perp}|i}, \tag{10}$$

$$\begin{aligned} \overset{\star}{\mathcal{L}}_{\bar{\mathbf{n}}_t} T_{(t)j\bar{\perp}} \;\; = \;\; & \Big(\bar{K}_t - 2\frac{\bar{N}_{t,\bar{\perp}}}{\bar{N}_t} - \frac{1}{c\bar{N}_t}\Big[\frac{1}{t} + \frac{\bar{N}_{t,t}}{\bar{N}_t}\Big]\Big) T_{(t)j\bar{\perp}} - c^{-2} \bar{a}_{\mathcal{F}j} T_{(t)\bar{\perp}\bar{\perp}} \\ & + c^{-2} \bar{a}_{\mathcal{F}i} \hat{T}^{i}_{(t)j} + c^{-2} \bar{a}_{\mathcal{F}j} \frac{t_0}{t} \frac{\bar{N}^i}{\bar{N}_t} T_{(t)i\bar{\perp}} - \hat{T}^{i}_{(t)j|i}, \end{aligned} \tag{11}$$

where $\overset{\star}{\mathcal{L}}_{\bar{\mathbf{n}}_t}$ denotes Lie derivative of spatial objects in the direction normal to the FHSs and '|' denotes spatial covariant derivation. (A "hat" denotes an object intrinsic to the FHSs.) See [4] for a derivation of these equations. Also postulated in [4] are the field equations (with $\kappa \equiv \frac{8\pi G}{c^4}$)

$$2\bar{R}_{(t)\bar{\perp}\bar{\perp}} = 2(c^{-4}\bar{a}_{\mathcal{F}k}\bar{a}^k_{\mathcal{F}} + c^{-2}\bar{a}^k_{\mathcal{F}|k} - \bar{K}_{(t)ik}\bar{K}^{ik}_{(t)} + \mathcal{L}_{\bar{\mathbf{n}}_t}\bar{K}_t) = \kappa(T_{(t)\bar{\perp}\bar{\perp}} + \hat{T}^i_{(t)i}), \tag{12}$$

$$\bar{R}_{(t)j\bar{\perp}} = \bar{K}^i_{(t)j|i} - \bar{K}_{t,j} = \kappa T_{(t)j\bar{\perp}}, \tag{13}$$

where $\bar{\mathbf{R}}_t$ is the Ricci tensor family and $\bar{\mathbf{K}}_t$ is the extrinsic curvature tensor family of the FHSs corresponding to the metric family (1). ($\bar{K}_t$ is the trace of $\bar{\mathbf{K}}_t$.) Notice that these field equations contain only one dynamical degree of freedom coupled explicitly to matter (i.e., gravity is essentially *scalar* in $(\mathcal{N}, \bar{\mathbf{g}}_t)$). However, by construction a second, *implicit* dynamical degree of freedom is added under the transformation $\bar{\mathbf{g}}_t \rightarrow \mathbf{g}_t$ [3, 4]. It should also be emphasized that, although the field equations are postulated rather than derived, they are by no means arbitrary; equation (12) for example, follows naturally from a geometrical correspondence with Newton-Cartan theory. Besides, in contrast to General Relativity, these field equations represent only a *partial* coupling between the space-time geometry and $\mathbf{T}_t$. Therefore, somewhat similar to Newtonian theory, the field equations (12), (13) are in principle quite independent of the local conservation laws (9) (and the equations of motion (7)). But this means that postulating the field equations directly is in principle not in any way inferior to "deriving" them from a postulated invariant action principle.

Finally, it is convenient to have expressions for the geometry intrinsic to the FHSs obtained from equation (1). We may straightforwardly derive the formulae

$$\bar{H}_{(t)ij} = c^{-2}\Big(\bar{a}^k_{\mathcal{F}|k} - \frac{1}{(\bar{N}_t t)^2}\Big)\bar{h}_{(t)ij} - c^{-4}\bar{a}_{\mathcal{F}i}\bar{a}_{\mathcal{F}j} - c^{-2}\bar{a}_{\mathcal{F}i|j}, \tag{14}$$

$$\bar{P}_t = \frac{6}{(c\bar{N}_t t)^2} + 2c^{-4}\bar{a}_{\mathcal{F}i}\bar{a}^i_{\mathcal{F}} - 4c^{-2}\bar{a}^i_{\mathcal{F}|i}, \tag{15}$$

where $\bar{\mathbf{h}}_t$ is the metric family intrinsic to the FHSs, $\bar{\mathbf{H}}_t$ is the Einstein tensor family intrinsic to the FHSs and $\bar{P}_t$ is the Ricci scalar family intrinsic to the FHSs in $(\mathcal{N}, \bar{\mathbf{g}}_t)$.

2.2 Special equations of motion

In this paper we analyze the equations of motion (7) in the case of a uniformly expanding, isotropic gravitational field in vacuum exterior to an isolated, spherically symmetric

source in an isotropic, compact spatial background. We also require that the source is at rest with respect to some GTCS. Furthermore we require that $\bar{N}_t$ is independent of x^0 (and t); i.e., that the only explicit time dependence is via t in the spatial scale factor (using the chosen GTCS). Then it turns out that also the FOs must be at rest with respect to the chosen GTCS and consequently the shift vector field vanishes. We denote this a "metrically static" case. This scenario may be taken as a generalization of the analogous case with a Minkowski background (that case is analyzed in [4]) and is more realistic since the Minkowski background is not a part of our theory but rather invoked as an approximation being useful in particular cases.

We start by making a specific *ansatz* for the form of $\bar{\mathbf{g}}_t$. Introducing a spherical GTCS $\{x^0, r, \theta, \phi\}$, where r is a Schwarzschild radial coordinate, we assume that the metric families $\bar{\mathbf{g}}_t$ and $\mathbf{g}_t$ can be written in a form compatible with equation (1) (using the notation $' \equiv \frac{\partial}{\partial r}$), i.e.,

$$
\begin{aligned}
c^2\overline{d\tau}_t^2 &= \bar{B}(r)(dx^0)^2 - (\frac{t}{t_0})^2\Big(\bar{A}(r)dr^2 + r^2 d\Omega^2\Big),\\
c^2 d\tau_t^2 &= B(r)(dx^0)^2 - (\frac{t}{t_0})^2\Big(A(r)dr^2 + r^2 d\Omega^2\Big),
\end{aligned} \tag{16}
$$

$$
\bar{A}(r) \equiv \frac{\Big[1 - r\frac{\bar{B}'(r)}{2\bar{B}(r)}\Big]^2}{1 - \frac{r^2}{\bar{B}(r)\Xi_0^2}}, \tag{17}
$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$ and $\Xi_0 \equiv ct_0$. Note that the spatial coordinate system covers only half of $\mathbf{S}^3$, thus the range of the radial coordinate is $r < \Xi_0$ only. The function $\bar{B}(r)$ may be calculated from the field equations; we treat this problem in the next section. The functions $A(r)$ and $B(r)$ may then be found from $\bar{\mathbf{g}}_t$ and $\bar{y}_t$.

We now calculate the metric connection coefficients from the metric family $\mathbf{g}_t$ given in equation (16). A straightforward calculation yields

$$
\begin{aligned}
\Gamma^r_{(t)rr} &= \frac{A'(r)}{2A(r)}, \qquad \Gamma^r_{(t)\theta\theta} = -\frac{r}{A(r)}, \qquad \Gamma^r_{(t)\phi\phi} = \Gamma^r_{(t)\theta\theta}\sin^2\theta,\\
\Gamma^r_{(t)00} &= (\frac{t_0}{t})^2\frac{B'(r)}{2A(r)}, \qquad \Gamma^\theta_{(t)r\theta} = \Gamma^\theta_{(t)\theta r} = \frac{1}{r}, \qquad \Gamma^\theta_{(t)\phi\phi} = -\sin\theta\cos\theta,\\
\Gamma^\phi_{(t)r\phi} &= \Gamma^\phi_{(t)\phi r} = \frac{1}{r}, \qquad \Gamma^\phi_{(t)\phi\theta} = \Gamma^\phi_{(t)\theta\phi} = \cot\theta, \qquad \Gamma^0_{(t)0r} = \Gamma^0_{(t)r0} = \frac{B'(r)}{2B(r)}.
\end{aligned} \tag{18}
$$

In the following we use the equations of motion (7) to find the paths of inertial test particles moving in the metric family $\mathbf{g}_t$. Since $\mathbf{a}_t$ vanishes for inertial test particles we

get the relevant equations by using equation (6) and inserting the expressions (18) into equation (7). This yields (making explicit use of the fact that $cdt = dx^0$ in a GTCS)

$$\frac{d^2r}{d\lambda^2}+\frac{A'(r)}{2A(r)}\Big(\frac{dr}{d\lambda}\Big)^2-\frac{r}{A(r)}\Big[\Big(\frac{d\theta}{d\lambda}\Big)^2+\sin^2\theta\Big(\frac{d\phi}{d\lambda}\Big)^2\Big]$$
$$+(\frac{t_0}{t})^2\frac{B'(r)}{2A(r)}\Big(\frac{dx^0}{d\lambda}\Big)^2+\frac{1}{ct}\frac{dr}{d\lambda}\frac{dx^0}{d\lambda}=0, \tag{19}$$

$$\frac{d^2\theta}{d\lambda^2}+\frac{2}{r}\frac{d\theta}{d\lambda}\frac{dr}{d\lambda}-\sin\theta\cos\theta\Big(\frac{d\phi}{d\lambda}\Big)^2+\frac{1}{ct}\frac{d\theta}{d\lambda}\frac{dx^0}{d\lambda}=0, \tag{20}$$

$$\frac{d^2\phi}{d\lambda^2}+\frac{2}{r}\frac{d\phi}{d\lambda}\frac{dr}{d\lambda}+2\cot\theta\frac{d\phi}{d\lambda}\frac{d\theta}{d\lambda}+\frac{1}{ct}\frac{d\phi}{d\lambda}\frac{dx^0}{d\lambda}=0, \tag{21}$$

$$\frac{d^2x^0}{d\lambda^2}+\frac{B'(r)}{B(r)}\frac{dx^0}{d\lambda}\frac{dr}{d\lambda}=0. \tag{22}$$

If we restrict the motion to the equatorial plane equation (20) becomes vacuous, and equation (21) reduces to

$$\frac{d^2\phi}{d\lambda^2}+\frac{2}{r}\frac{d\phi}{d\lambda}\frac{dr}{d\lambda}+\frac{1}{ct}\frac{d\phi}{d\lambda}\frac{dx^0}{d\lambda}=0. \tag{23}$$

Dividing equation (23) by $\frac{d\phi}{d\lambda}$ we find (assuming $\frac{d\phi}{d\lambda}\neq 0$)

$$\frac{d}{d\lambda}\Big[\ln\Big(\frac{d\phi}{d\lambda}\Big)+\ln\Big(r^2\frac{t}{t_0}\Big)\Big]=0. \tag{24}$$

We thus have a constant of the motion, namely

$$J\equiv\frac{t}{t_0}r^2\frac{d\phi}{d\lambda}. \tag{25}$$

Dividing equation (22) by $\frac{dx^0}{d\lambda}$ yields

$$\frac{d}{d\lambda}\Big[\ln\Big(\frac{dx^0}{d\lambda}\Big)+\ln B(r)\Big]=0. \tag{26}$$

Equation (26) yields a constant of the motion which we can absorb into the definition of λ such that a solution of equation (26) is [7]

$$\frac{dx^0}{d\lambda}=\frac{1}{B(r)}. \tag{27}$$

Multiplying equation (19) by $\frac{2t^2 A(r)}{t_0^2}\frac{dr}{d\lambda}$ and using the expressions (25), (27) we find

$$\frac{d}{d\lambda}\left[\frac{t^2 A(r)}{t_0^2}\Big(\frac{dr}{d\lambda}\Big)^2 - \frac{1}{B(r)} + \frac{J^2}{r^2}\right] = 0, \tag{28}$$

thus a constant E of the motion is defined by

$$\frac{t^2 A(r)}{t_0^2}\Big(\frac{dr}{d\lambda}\Big)^2 - \frac{1}{B(r)} + \frac{J^2}{r^2} \equiv -E. \tag{29}$$

Equation (29) may be compared to an analogous expression obtained for the static isotropic gravitational field in the metric framework [7]. Inserting the formulae (25), (27) and (29) into equation (16) and using the fact that in a GTCS we can formally write $dx^0 = cdt$ when traversing the family of metrics, we find

$$c^2 d\tau_t^2 = E d\lambda^2. \tag{30}$$

Thus our equations of motion (7) force $d\tau_t/d\lambda$ to be constant, quite similarly to the case when the total connection is metric, as in the metric framework. From equation (30) we see that we must have $E = 0$ for photons and $E > 0$ for material particles.

We may eliminate the parameter λ from equations (25), (27), (29) and (30) and alternatively use t as a time parameter. This yields

$$\frac{t}{t_0} r^2 \frac{d\phi}{cdt} = B(r)J, \tag{31}$$

$$(\frac{t}{t_0})^2 A(r)B^{-2}(r)\Big(\frac{dr}{cdt}\Big)^2 - \frac{1}{B(r)} + \frac{J^2}{r^2} \equiv -E, \tag{32}$$

$$d\tau_t^2 = EB^2(r)dt^2. \tag{33}$$

We may integrate equations (31) and (32) to find the time history $(r(t), \phi(t))$ along the curve if the functions $A(r)$ and $B(r)$ are known.

For the static vacuum metric case with no global NKE, one can solve the geodesic equation for particles orbiting in circles with different radii, and from this find the asymptotically Keplerian nature of the corresponding rotational curve [7]. In our case we see from equations (31) and (32) that we can find circle orbits as solutions; such orbits have the property that the orbital speed $w(r) = B^{-1/2}(r)\frac{t}{t_0} r \frac{d\phi}{dt}$ is independent of t. (Here $w(r)$ is the norm of the coordinate 3-velocity $\mathbf{w}_t = \sqrt{1 - \frac{w^2}{c^2}}\frac{d\phi}{d\tau_t}\frac{\partial}{\partial\phi}$.)

3 Metrically static, spherically symmetric systems

3.1 Perfect fluid sources

We now seek general solutions of the type (16) of the field equations, and where the source is modelled as a perfect fluid. Then the active stress-energy tensor $\mathbf{T}_t$ takes the form

$$\mathbf{T}_t = (\tilde{\rho}_{\mathrm{m}} + \tilde{p}/c^2)\bar{\mathbf{u}}_t{\otimes}\bar{\mathbf{u}}_t + \tilde{p}\bar{\mathbf{g}}_t, \tag{34}$$

where $\tilde{\rho}_{\mathrm{m}}$ is the density of active mass-energy and $\tilde{p}$ is the active pressure seen in the local rest frame of the fluid. Moreover $\bar{\mathbf{u}}_t$ is the 4-velocity of the fluid in $(\mathcal{N}, \bar{\mathbf{g}}_t)$. But what can be measured locally is not $\mathbf{T}_t$ but *the passive stress-energy tensor* $\bar{\mathcal{T}}_t$ in $(\mathcal{N}, \bar{\mathbf{g}}_t)$, given by

$$\bar{\mathcal{T}}_t = (\rho_{\mathrm{m}} + p/c^2)\bar{\mathbf{u}}_t{\otimes}\bar{\mathbf{u}}_t + p\bar{\mathbf{g}}_t, \tag{35}$$

where ρ_{m} is the passive mass-energy as measured in the local rest frame of the fluid and p is the associated passive pressure. Note that the counterpart $\mathcal{T}_t$ in $(\mathcal{N}, \mathbf{g}_t)$ of $\bar{\mathcal{T}}_t$ is given by

$$\mathcal{T}_t = \sqrt{\frac{\bar{h}_t}{h_t}}\Big[(\rho_{\mathrm{m}} + p/c^2)\mathbf{u}_t{\otimes}\mathbf{u}_t + p\mathbf{g}_t\Big], \tag{36}$$

where $\bar{h}_t$ and h_t are the determinants of the spatial metrics $\bar{\mathbf{h}}_t$ and $\mathbf{h}_t$, respectively. Besides, the relationship between $\tilde{\rho}_{\mathrm{m}}$ and ρ_{m} is given by

$$\tilde{\rho}_{\mathrm{m}} = \begin{cases} \frac{t}{t_0}\bar{N}_t\rho_{\mathrm{m}} & \text{for a fluid of material particles,} \\ \frac{t^2}{t_0^2}\bar{N}_t^2\rho_{\mathrm{m}} & \text{for the electromagnetic field,} \end{cases} \tag{37}$$

and similarly for the relationship between $\tilde{p}$ and p. In the following sections we set up the relevant equations both for the interior and the exterior gravitational field. As we shall see the equations valid inside the source get quite complex; this makes analytical calculations rather impracticable so the equations should be solved numerically. However, we have not performed any numerical calculations. On the other hand an exact solution may be found for the exterior field.

3.2 The interior field

In this section we analyze the gravitational field inside the source. That is, we do the necessary analytical calculations in order to write the relevant equations in a form ap-

propriate for numerical treatment. Proceeding with this, from the definitions we get

$$\begin{aligned}
c^{-2}\bar{a}_{\mathcal{F}r} &= \frac{\bar{B}'}{2\bar{B}}, \qquad c^{-2}\bar{a}_{\mathcal{F}r|r} = \frac{\bar{B}''}{2\bar{B}} - \frac{1}{2}\Big(\frac{\bar{B}'}{\bar{B}}\Big)^2 - \frac{\bar{A}'\bar{B}'}{4\bar{A}\bar{B}}, \\
c^{-2}\sin^{-2}\theta\bar{a}_{\mathcal{F}\phi|\phi} &= c^{-2}\bar{a}_{\mathcal{F}\theta|\theta} = \frac{r\bar{B}'}{2\bar{A}\bar{B}}, \qquad \bar{P}_t = (\frac{t_0}{t})^2\frac{2}{\bar{A}}\Big(r^{-2}(\bar{A}-1) + \frac{\bar{A}'}{r\bar{A}}\Big),
\end{aligned} \tag{38}$$

$$\begin{aligned}
c^{-2}\bar{a}^k_{\mathcal{F}|k} &= (\frac{t_0}{t})^2\Big(\frac{\bar{B}''}{2\bar{A}\bar{B}} - \frac{1}{2\bar{A}}\Big(\frac{\bar{B}'}{\bar{B}}\Big)^2 - \frac{\bar{A}'\bar{B}'}{4\bar{A}^2\bar{B}} + \frac{\bar{B}'}{r\bar{A}\bar{B}}\Big), \\
\bar{H}_{(t)rr} &= r^{-2}(1-\bar{A}), \qquad \sin^{-2}\theta\bar{H}_{(t)\phi\phi} = \bar{H}_{(t)\theta\theta} = -\frac{r\bar{A}'}{2\bar{A}^2}.
\end{aligned} \tag{39}$$

Now active mass density varies formally as $\bar{F}_t^{-2}$ when measured in atomic units. For reasons of convenience we choose to extract this formal variability explicitly. What is left after separating out the formal variability from the active mass density is by definition *the coordinate volume density of active mass* $\bar{\rho}_{\rm m}$. (The corresponding pressure is $\bar{p}$.) For the case when the perfect fluid is co-moving with the FOs (i.e., $T_{(t)\bar{\perp}j} = 0$) we find from equation (34)

$$T_{(t)\bar{\perp}\bar{\perp}} = \tilde{\rho}_{\rm m}c^2 \equiv \frac{t_0^2}{t^2}\frac{\bar{\rho}_{\rm m}c^2}{\bar{B}}, \qquad T^r_{(t)r} = T^\theta_{(t)\theta} = T^\phi_{(t)\phi} = \tilde{p} \equiv \frac{t_0^2}{t^2}\frac{\bar{p}}{\bar{B}}. \tag{40}$$

Furthermore, using equations (40) the local conservation laws (10), (11) applied to the metrically static case yield (with $\dot{} \equiv \frac{\partial}{\partial t}$)

$$\dot{\bar{\rho}}_{\rm m} = \dot{\bar{p}} = 0, \qquad \bar{p}' = -c^{-2}\bar{a}_{\mathcal{F}r}(\bar{\rho}_{\rm m}c^2 - 3\bar{p}) = -(\bar{\rho}_{\rm m}c^2 - 3\bar{p})\frac{\bar{B}'}{2\bar{B}}. \tag{41}$$

Equations (41) are valid for any metrically static perfect fluid. But to have experimental input we need to specify an equation of state $p = p(\rho_{\rm m})$ consistent with the metrically static condition. That is, equations (41) are valid only when the explicit dependence $p(\rho_{\rm m})$ takes the form $p \propto \rho_{\rm m}$ since otherwise the equation of state will not be consistent with the given time evolution. Moreover, once a suitable equation of state is given it is necessary to use the expressions (37) and (40) relating $\bar{\rho}_{\rm m}$ to the passive mass density $\rho_{\rm m}$ and similarly for a relationship between $\bar{p}$ and the passive pressure p.

To find how the active mass m_t varies in space-time, note that we are free to choose the background value of the active mass far from the source to be m_0. We use this to define G as the constant measured in a Cavendish experiment far from the source at epoch t_0. Then, using equations (8) and (38) we find

$$m_t(r,t) = \bar{B}^{1/2}(r)\frac{t}{t_0}m_0. \tag{42}$$

Now we can insert the equations (38), (39) and (40) into the field equation (12). Since $\bar{\mathbf{K}}_t$ vanishes identically for the metrically static case [4], equations (13) become vacuous and equation (12) yields

$$\frac{\bar{B}''}{\bar{B}} - \frac{1}{2}\Big(\frac{\bar{B}'}{\bar{B}}\Big)^2 - \frac{\bar{A}'\bar{B}'}{2\bar{A}\bar{B}} + \frac{2\bar{B}'}{r\bar{B}} = \kappa\frac{\bar{A}}{\bar{B}}(\bar{\rho}_{\rm m}c^2 + 3\bar{p}). \tag{43}$$

A second equation can be found from the equation (15) for the spatial curvature. The radial component just gives equation (17) again but the angular components yield a new equation, which can be combined with equation (43) to give

$$\frac{\bar{A}'}{\bar{A}} - \frac{\bar{B}'}{\bar{B}} - \frac{r}{2}\Big(\frac{\bar{B}'}{\bar{B}}\Big)^2 - \frac{2r\bar{A}}{\Xi_0^2\bar{B}} = -r\kappa\frac{\bar{A}}{\bar{B}}(\bar{\rho}_{\rm m}c^2 + 3\bar{p}). \tag{44}$$

We may now eliminate $\bar{A}(r)$ and $\bar{A}'(r)$ from equation (43) by inserting equations (17) and (44). The result is

$$\Big(1 - \frac{r^2}{\bar{B}(r)\Xi_0^2}\Big)\frac{\bar{B}''(r)}{\bar{B}(r)} \quad - \quad \Big(1 - \frac{2r^2}{\bar{B}(r)\Xi_0^2}\Big)\Big(\frac{\bar{B}'(r)}{\bar{B}(r)}\Big)^2 + \frac{2}{r}\Big(1 - \frac{3r^2}{2\bar{B}(r)\Xi_0^2}\Big)\frac{\bar{B}'(r)}{\bar{B}(r)}$$
$$-\frac{r}{4}\Big(\frac{\bar{B}'(r)}{\bar{B}(r)}\Big)^3 = \frac{\kappa}{\bar{B}(r)}\Big(1 - \frac{r\bar{B}'(r)}{2\bar{B}(r)}\Big)^3\Big[\bar{\rho}_{\rm m}(r)c^2 + 3\bar{p}(r)\Big]. \tag{45}$$

To solve equation (45) numerically for $\bar{B}(r)$ one may proceed as follows. First specify the boundary conditions at the center of the body. From equation (17) we easily see that $\bar{A}(0) = 1$, and moreover we must have $\bar{B}'(0) = \bar{p}'(0) = 0$, so $\bar{A}'(0) = 0$ from equation (44). Furthermore, noting that $r^{-1}\bar{B}'$ must be stationary near the center of the body we have

$$\bar{B}''(0) = \lim_{r\rightarrow 0}\Big[r^{-1}\bar{B}'(r)\Big], \qquad \Rightarrow \qquad \bar{B}''(0) = \frac{\kappa}{3}\Big(\bar{\rho}_{\rm m}(0)c^2 + 3\bar{p}(0)\Big), \tag{46}$$

where the implication follows from equation (43).

To specify any particular model, choose the boundary condition $\bar{p}(0)$ at some arbitrary time. Also choose some arbitrary value $\bar{B}(0)$ as an initial value for iteration. It must be possible to check how well the chosen $\bar{B}(0)$ reproduces the boundary condition for $\bar{B}(\mathcal{R})$ at the surface $r = \mathcal{R}$ of the body. That is, to match the exterior (vacuum) solution we must have (see the next section, equation (54))

$$\bar{B}(\mathcal{R}) = 1 - \frac{r_{\rm s0}}{\mathcal{R}} + \frac{r_{\rm s0}^2}{2\mathcal{R}^2} + \frac{r_{\rm s0}\mathcal{R}}{2\Xi_0^2} - \frac{r_{\rm s0}^3}{8\mathcal{R}^3} + \cdots, \tag{47}$$

where $r_{\rm s0}$ is a constant which may be identified with the Schwarzschild radius of the body at the arbitrary time t_0. Hence, by definition we set

$$r_{\rm s0} \equiv \frac{2M_{t_0}G}{c^2}, \quad M_t \equiv c^{-2}\int\!\!\int\!\!\int \bar{N}_t\Big[T_{(t)\bot\bot} + \hat{T}^i_{(t)i}\Big]d\bar{V}_t = 4\pi\frac{t}{t_0}\int_0^{\mathcal{R}}\frac{\sqrt{\bar{A}}}{\sqrt{\bar{B}}}\Big[\bar{\rho}_{\rm m} + 3\bar{p}/c^2\Big]r^2dr, \tag{48}$$

where the integration is taken over the whole body. (The particular form of M_t follows directly from equation (12) applied to the interior of a spherically symmetric, metrically static source when extrapolated to the exact exterior solution given by equations (52) and (53) below.) Note that M_{t_0} is defined to be the total (active) mass-energy of the body as measured by distant orbiters at epoch t_0. Notice in particular that this expression depends not only on the mass density of the body but also on the pressure.

As already mentioned; to have a metrically static system it is necessary to specify an equation of state of the type $p \propto \rho_{\rm m}$ (potential implicit dependences not included) since this ensures that $\bar{\rho}_{\rm m}$ and $\bar{p}$ are independent of t. Since the equation of state for an ideal gas has the required form and since non-degenerate star matter is reasonably well approximated by an ideal gas, it should not be too unrealistic to apply the metrically static condition to main sequence stars. (The metrically static condition holds in general for an ideal gas even if the gas is not isothermal; i.e., even if the temperature depends on the pressure so that the equation of state takes a polytropic form.) Once a suitable equation of state has been specified we may use equations (37) and (40) to find $\bar{\rho}_{\rm m}$ and $\bar{p}$ and then integrate equation (45) outwards from $r = 0$ using equation (41) until the pressure vanishes. The surface of the body is now reached (that is $r = \mathcal{R}$). If the calculated value $\bar{B}(\mathcal{R})$ does not match the boundary condition (47), add a constant to $\bar{B}$ everywhere and repeat the calculation with the new value of $\bar{B}(0)$. Iterate until sufficient accuracy is achieved.

Once we have done the above calculations for an arbitrary time we know the time evolution of the system from equation (41). That is, a spherical gravitationally bound body made of perfect fluid obeying an equation of state of type $p \propto \rho_{\rm m}$ will expand according to the Hubble law. But for bodies made of perfect fluid obeying other equations of state (degenerate star matter for example) the expansion may induce instabilities; mass currents will be set up and such systems cannot be metrically static. However, for the metrically static case the gravitational field interior to the body will expand along with the fluid; this is similar to the expansion of the exterior gravitational field found in the next section. We are now able to calculate the family $\bar{\mathbf{g}}_t$ inside the body. To find the corresponding family $\mathbf{g}_t$ one uses the method described in [3, 4].

We finish this section by estimating how the cosmic expansion will affect a spherically symmetric body made of perfect fluid obeying an equation of state of the form $p \propto \rho_{\rm m}^{\gamma}$ (e.g., a polytrope made of degenerate matter). To do that, we assume that the hydrodynamical effects on the gravitational field coming from instabilities can be neglected; i.e., we assume that the body can be treated as being approximately in hydrostatic equilibrium for each epoch t. (Effects coming from gravitational heating of the body due to contraction are

also neglected.) To justify this approximation we work in the Newtonian limit. That is, we first take the Newtonian limits of equations (41) and (45), getting

$$\frac{d}{dr}\frac{r^2}{\bar{\rho}_{\mathrm{m}}}\bar{p}' = -4\pi G r^2 \bar{\rho}_{\mathrm{m}0}, \tag{49}$$

where $\bar{\rho}_{\mathrm{m}0}$ is the density field $\bar{\rho}_{\mathrm{m}}$ at epoch t_0. As it stands, equation (49) is valid only for the fixed epoch t_0 if $\gamma \neq 1$. However, by making the substitution $r \rightarrow (\frac{t_0}{t})^{\frac{1}{3\gamma-4}} r \equiv \bar{r}$, $\bar{\rho}_{\mathrm{m}0} \rightarrow (\frac{t}{t_0})^{\frac{3}{3\gamma-4}} \bar{\rho}_{\mathrm{m}0} \equiv \bar{\rho}_{\mathrm{m}}$, $G \rightarrow \frac{t}{t_0} G \equiv G_t$, equation (49) may effectively be transformed from epoch t_0 to epoch t and it may then be applied to the body at each fixed epoch t even if $\gamma \neq 1$. Equation (49) then becomes equivalent to its counterpart in Newtonian theory except for a variable G_t. Thus the usual Newtonian analysis of polytropes [7] applies, but with G_t variable. And as consequences of this we see that the physical radius of a polytrope will actually *shrink* with epoch (if $\gamma > \frac{4}{3}$), and that the Chandrasekhar mass limit will decrease with epoch. Thus any white dwarf made of degenerate matter is predicted to shrink with epoch and eventually explode as a type Ia supernova when the Chandrasekhar mass gets close to the mass of the white dwarf. In particular this should happen to isolated white dwarfs, so according to quasi-metric theory it is not necessary to invoke mass accretion from exterior sources to ignite type Ia supernovae.

3.3 The exterior field

In this section we write down one equation which must be solved to find the unknown function $\bar{B}(r)$ valid for the exterior gravitational field. We give an exact solution of this equation and use this solution to find exact expressions for the functions $\bar{A}(r)$, $A(r)$ and $B(r)$ as well. Now, from equations (3), (16) and (38) we get (since $\bar{N}_t = \sqrt{\bar{B}(r)}$)

$$\frac{\partial \bar{h}_{(t)ij}}{\partial t} = \frac{2}{t}\bar{h}_{(t)ij}, \qquad y_t(t,r) = \frac{t_0}{t}\frac{c\bar{B}'(r)}{2\bar{B}(r)\sqrt{\bar{A}(r)}}. \tag{50}$$

Furthermore, equation (45) without source terms reads

$$\begin{aligned}\Big(1 - \frac{r^2}{\bar{B}(r)\Xi_0^2}\Big)\frac{\bar{B}''(r)}{\bar{B}(r)} \quad &- \quad \Big(1 - \frac{2r^2}{\bar{B}(r)\Xi_0^2}\Big)\Big(\frac{\bar{B}'(r)}{\bar{B}(r)}\Big)^2 \\ &+ \quad \frac{2}{r}\Big(1 - \frac{3r^2}{2\bar{B}(r)\Xi_0^2}\Big)\frac{\bar{B}'(r)}{\bar{B}(r)} - \frac{r}{4}\Big(\frac{\bar{B}'(r)}{\bar{B}(r)}\Big)^3 = 0.\end{aligned} \tag{51}$$

Notice that this equation may be viewed as a perturbation of the situation we get if we take the limit $\Xi_0 \rightarrow \infty$. That case was analyzed in [4], and it was found that the solution is unique. Moreover, it turns out that the solution of equation (51) is unique as well.

Before we try to solve equation (51), it is important to notice that no solution of it can exist on a whole FHS (except the trivial solution $\bar{B}$ =constant), according to the maximum principle applied to a closed Riemannian 3-manifold. The reason for this is the particular form of the field equations, see reference [4] and references therein for justification. This means that *in quasi-metric theory, isolated systems cannot exist except as an approximation.*

Even if a non-trivial solution of equation (51) does not exist on a whole FHS we may try to find a solution valid in some finite region of a FHS. That is, we want to find a solution in the region $\mathcal{R} < r < \Xi_0$, where $\mathcal{R}$ is the coordinate radius of the source. To get a unique solution of equation (51) it is sufficient to require that this solution has the correct correspondence in the limit $\Xi_0 \to \infty$. But what happens at the boundary $\Xi_0 \equiv ct_0$ is not very interesting anyway since the approximation made by assuming an isolated system is physically reliable only if $\frac{r}{\Xi_0} \ll 1$.

To find an exact solution of equation (51) it is useful to rewrite it in terms of an isotropic radial coordinate. One then finds the following exact solution (transformed back as a function of r)

$$\bar{B}(r) = \Big(\sqrt{1 + (\frac{r_{\rm s0}}{2r})^2 - \frac{r^2}{\Xi_0^2}} - \frac{r_{\rm s0}}{2r}\Big)^2 + \frac{r^2}{\Xi_0^2}, \tag{52}$$

which may be inserted into equation (51) to check that it really is a solution. Moreover, from equations (52) and (16) we find

$$\bar{A}(r) = \Big[1 + (\frac{r_{\rm s0}}{2r})^2 - \frac{r^2}{\Xi_0^2}\Big]^{-1} \bar{B}(r). \tag{53}$$

For small r we may write expressions (52) and (53) as series expansions, i.e., as perturbations around the analogous problem in a Minkowski background. But in contrast to the analogous case with a Minkowski background there exists the extra scale Ξ_0 in addition to the Schwarzschild radius $r_{\rm s0}$ defined in equation (48). To begin with we try to model the gravitational field exterior to galactic-sized objects, so we may assume that the typical scales involved are determined by $\frac{r}{\Xi_0} \gtrsim \frac{r_{\rm s0}}{r}$; this criterion tells how to compare the importance of the different terms of the series expansion. One may straightforwardly show that series expansions of equations (52) and (53) yield

$$\begin{aligned} \bar{B}(r) &= 1 - \frac{r_{\rm s0}}{r} + \frac{r_{\rm s0}^2}{2r^2} + \frac{r_{\rm s0} r}{2\Xi_0^2} - \frac{r_{\rm s0}^3}{8r^3} + \cdots, \qquad \Rightarrow \\ \bar{A}(r) &= 1 - \frac{r_{\rm s0}}{r} + \frac{r_{\rm s0}^2}{4r^2} + \frac{r^2}{\Xi_0^2} + \cdots. \end{aligned} \tag{54}$$

To construct the family $\mathbf{g}_t$ as described in [3] we need the quantity $v(r)$, which for spherically symmetric systems takes the form [4]

$$v(r) = \bar{y}_t r\sqrt{\bar{h}_{(t)rr}} = \frac{cr}{2}\frac{\bar{B}'(r)}{\bar{B}(r)} = \frac{r_{\rm s0}}{2r}\frac{c}{\sqrt{1+(\frac{r_{\rm s0}}{2r})^2 - \frac{r^2}{\Xi_0^2}}} = \frac{r_{\rm s0}c}{2r}[1+O(\frac{r_{\rm s0}^2}{r^2})]. \tag{55}$$

We note that $v(r)$ does not depend on t. The functions $A(r)$ and $B(r)$ are found from the relations (valid for spherically symmetric systems [4])

$$A(r) = \Big(\frac{1+\frac{v(r)}{c}}{1-\frac{v(r)}{c}}\Big)^2\bar{A}(r), \qquad B(r) = \Big(1-\frac{v^2(r)}{c^2}\Big)^2\bar{B}(r). \tag{56}$$

From equations (52), (53) and (56) we then get

$$B(r) = \frac{(1-\frac{r^2}{\Xi_0^2})^2}{(1+(\frac{r_{\rm s0}}{2r})^2-\frac{r^2}{\Xi_0^2})^2}\Big[\Big(\sqrt{1+(\frac{r_{\rm s0}}{2r})^2-\frac{r^2}{\Xi_0^2}}-\frac{r_{\rm s0}}{2r}\Big)^2+\frac{r^2}{\Xi_0^2}\Big], \tag{57}$$

$$A(r) = \frac{\Big(\sqrt{1+(\frac{r_{\rm s0}}{2r})^2-\frac{r^2}{\Xi_0^2}}+\frac{r_{\rm s0}}{2r}\Big)^2}{1+(\frac{r_{\rm s0}}{2r})^2-\frac{r^2}{\Xi_0^2}}\Big\{1+\frac{\Big(\sqrt{1+(\frac{r_{\rm s0}}{2r})^2-\frac{r^2}{\Xi_0^2}}+\frac{r_{\rm s0}}{2r}\Big)^2}{\Big(1-\frac{r^2}{\Xi_0^2}\Big)^2}\frac{r^2}{\Xi_0^2}\Big\}. \tag{58}$$

Notice that although $B(r)$ increases for small r, for large r it eventually reaches a maximum value and then decreases towards zero when $r{\rightarrow}\Xi_0$. This is merely a curious effect due to the global curvature of space and the unrealistic assumption that an isolated source determines the gravitational field at cosmological distances. That is, it is utterly unrealistic to assume that an isolated source dominates the gravitational field over cosmological scales and that this source has been present since the beginning of time. Thus the from equation (57) inferred gravitational repulsion on cosmological scales is nothing but an unrealistic model artefact.

It is useful to have series expansions for $B(r)$ and $A(r)$. Putting these into a family of line elements we find

$$\begin{aligned} ds_t^2 &= -\Big(1-\frac{r_{\rm s0}}{r}+\frac{r_{\rm s0}r}{2\Xi_0^2}+\frac{3r_{\rm s0}^3}{8r^3}+\cdots\Big)(dx^0)^2 \\ &\quad +(\frac{t}{t_0})^2\Big(\Big\{1+\frac{r_{\rm s0}}{r}+\frac{r^2}{\Xi_0^2}+\frac{r_{\rm s0}^2}{4r^2}+\cdots\Big\}dr^2+r^2d\Omega^2\Big). \end{aligned} \tag{59}$$

This expression represents the wanted metric family as a series expansion. Note in particular the fact that all spatial dimensions expand whereas the corresponding Newtonian potential $-U = -\frac{c^2 r_{\rm s0}}{2r}$ (to Newtonian order) remains constant for a fixed FO. This means

that the true radius of any circle orbit (i.e., with r constant) increases but such that the orbital speed remains constant. That is, *the (active) mass of the central object as measured by distant orbiters increases to exactly balance the effect on circle orbit velocities of expanding circle radii.* This is not as outrageous as it may seem due to the extra formal variation of atomic units built into our theory. So this result is merely a consequence of the fact that the coupling between matter and geometry depends directly on the formal variation via the field equations.

What is measured by means of distant orbiters is not the "bare" mass M_t itself but rather the combination M_tG. We have, however, *defined* G to be a constant. And as might be expected, it turns out that the variation of M_tG with t as inferred from equation (59) is exactly that found directly from the formal variation of the active mass M_t with t by using equations (8) and (48). This means that the dynamically measured mass increase should not be taken as an indication of actual particle creation but that the general dynamically measured mass scale should be taken to change via a linear increase of M_t with t, and that this is directly reflected in the gravitational field of the source. That is, measured in atomic units active mass increases linearly with epoch in accordance with equation (48).

The dynamical measurement of the mass of the central object by means of distant orbiters does not represent a local test experiment. Nevertheless the dynamically measured mass increase thus found is just as "real" as the expansion in the sense that neither should be neglected on extended scales. This must be so since in quasi-metric relativity, the global scale increase and the dynamically measured mass increase are two different aspects of the same basic phenomenon.

4 The effects of cosmic expansion on gravitation

4.1 Shapes of orbits and rotational curves

We now explore which kinds of free-fall orbits we get from equation (59) and the equations of motion. To begin with we find the shape of the rotational curve as defined from the coordinate 3-velocities $\mathbf{w}_t$ of the circle orbits. (The 4-velocities $\mathbf{u}_t$ may be split up into pieces respectively orthogonal to and intrinsic to the FHSs according to the formula $\sqrt{1-\frac{w^2}{c^2}}\mathbf{u}_t = c\mathbf{n}_t + \mathbf{w}_t$.) Since equation (32) has no time dependence for such orbits we can do a standard calculation [7] and the result is that orbital speed w varies as

$$w(r) = \frac{t}{t_0}r\frac{d\phi}{dt}\sqrt{1-\frac{w^2}{c^2}}\frac{dt}{d\tau_t} = B^{-1/2}(r)\frac{t}{t_0}r\frac{d\phi}{dt} = \sqrt{\frac{B'(r)r}{2B(r)}}c, \tag{60}$$

where the second step follows from the formula $\frac{d\tau_t}{dt} = \sqrt{B(r) - \frac{t^2}{t_0^2} r^2 (\frac{d\phi}{cdt})^2}$ (obtained from equation (16) for circular motion) together with a consistency requirement. However when we apply equation (60) to the metric family (59) we get a result essentially identical to the standard Keplerian rotational curve; the only effect of the dynamically measured mass increase and the non-kinematical expansion is to increase the scale but such that the shape of the rotational curve is unaffected. It is true that $B(r)$ as found from equation (59) contains a term linear in r in addition to terms falling off with increasing r; in reference [8] it is shown that such a linear term may be successfully used to model the asymptotically non-Keplerian rotational curves of spiral galaxies. But the numerical value of the linear term found from equation (59) is too small by a factor of order 10^{10} to be able to match the data. So at least the simple model considered in this paper is unable to explain the asymptotically non-Keplerian rotational curves of spiral galaxies from first principles.

Another matter is how the time dependence in the equations of motion will affect the time histories and shapes of more general orbits than the circle orbits. Clearly time histories will be affected as can be seen directly from equation (32). However to see if this is valid for shapes as well we may insert equation (31) into equation (32) to obtain r as a function of ϕ. This yields

$$\frac{A(r)}{r^4}\Big(\frac{dr}{d\phi}\Big)^2 + \frac{1}{r^2} - \frac{1}{J^2 B(r)} = -\frac{E}{J^2}, \tag{61}$$

and this is identical to the equation valid for the case of a single spherically symmetric static metric [7]. Thus the shapes of free-fall orbits are unaffected by the global non-kinematical expansion present in the metric family (59).

4.2 Expanding space and the solar system

If one neglects the gravitational effects of the galaxy, one may try to apply the metric family (59) to the solar system (by using it to describe the gravitational field of the Sun). But the solar system is not at rest with respect to the cosmic rest frame; this follows from the observed dipole in the cosmic microwave background radiation. Besides, if we identify the cosmic rest frame with some GTCS, the FOs will be at rest with respect to this GTCS only in the limit where the motions of local gravitational sources can be neglected. Consequently, if we use a GTCS identified with the cosmic rest frame, it is necessary to introduce a non-vanishing shift vector field to take into account the motion of the solar system barycenter. And expressed in such a GTCS the metric family describing the gravitational field of the Sun cannot be (approximately) metrically static since the

metric coefficients will now depend on x^0. However, as a good approximation, we may neglect the gravitational effects of the galaxy and treat the solar system as an isolated system moving with constant velocity with respect to the cosmic rest frame. In this case we are always able to find a GTCS in which the solar system barycenter is at rest and where the metric coefficients do not depend on x^0 by construction (when gravitational fields of other bodies than the Sun are neglected). Furthermore it turns out that the FOs must be at rest with respect to this new GTCS, so the shift vector field vanishes and all calculations we have done in this paper are still valid. Thus, if we treat the solar system as approximately isolated, we can neglect its motion with respect to the cosmic rest frame and use the metric family (59) to describe the gravitational field of the Sun [9]. Moreover, the solar system is so small that we can neglect any dependence on Ξ_0. The errors made by neglecting terms depending on Ξ_0 in equation (59) are insignificant since the typical scales involved for the solar system are determined by $\frac{r}{\Xi_0} \lesssim \frac{r_{s0}^3}{r^3}$. Equation (59) then takes the form

$$ds_t^2 = -\Big(1 - \frac{r_{s0}}{r} + O(\frac{r_{s0}^3}{r^3})\Big)(dx^0)^2 + (\frac{t}{t_0})^2\Big(\{1 + \frac{r_{s0}}{r} + O(\frac{r_{s0}^2}{r^2})\}dr^2 + r^2 d\Omega^2\Big). \tag{62}$$

Equation (61) shows that the shapes of orbits are unaffected by the expansion; this means that all the classical solar system tests come in just as for the analogous case of a Minkowski background [4]. However, we get at least one extra prediction (irrespective of whether or not the galactic gravitational field can be neglected); from equation (62) we see that the effective distance between the Sun and any planet is predicted to have been smaller in the past. That is, the spatial coordinates are co-moving rather than static, thus distances within the solar system and its gravitational field (measured in atomic units) should not be static with respect to the cosmic expansion. For example, the distance between the Sun and the Earth at the time of its formation may have been about 50% less than today. But since main sequence stars are predicted to expand according to quasi-metric theory, a small Earth-Sun distance should not be incompatible with paleo-climatic data, since the Sun is expected to have been smaller and thus dimmer in the past. Actually, since neither the temperature at the center of the Sun (as estimated from the virial theorem), nor the radiation energy gradient times the mean free path length of a photon depend on t, the cosmic luminosity evolution of the Sun should be determined from the cosmic expansion of its surface area as long as the ideal gas approximation is sufficient. And this luminosity evolution exactly balances the effects of an increasing Earth-Sun distance on the effective solar radiation received at the Earth.

However, an obvious question is if the predicted effect of the expansion on the time histories of non-relativistic orbits is compatible with the observed motions of the planets.

In order to try to answer this question it is illustrating to calculate how the orbit period of any planet depends on t. For simplicity consider a circular orbit $r = R$ =constant. Equation (31) then yields

$$\frac{d\phi}{dt} = \frac{t_0}{t} B(R) R^{-2} Jc. \tag{63}$$

Now integrate equation (63) one orbit period $T << t$ (i.e., from t to $t+T$). The result is

$$T(t) = t(\exp\Big[\frac{T_{\rm GR}}{t_0}\Big] - 1) = \frac{t}{t_0} T_{\rm GR}(1 + \frac{T_{\rm GR}}{2t_0} + \cdots), \qquad T_{\rm GR} \equiv \frac{2\pi R^2}{cJB(R)}, \tag{64}$$

where $T_{\rm GR}$ is the orbit period as predicted from General Relativity. From (64) we see that (sidereal) orbit periods are predicted to increase linearly with cosmic scale, i.e.

$$T(t) = \frac{t}{t_0} T(t_0), \qquad \frac{dT}{dt} = \frac{T(t_0)}{t_0}, \tag{65}$$

and such that any ratio between periods of different orbits remains constant. In particular equation (65) predicts that the (sidereal) year $T_{\rm E}$ should be increasing with about 2.5 ms/yr and the martian year $T_{\rm M}$ should be increasing by about 4.7 ms per martian year at the present epoch. This should be consistent with observations since the observed difference in the synodical periods of Mars and the Earth is accurate to about 5 ms.

To compare predictions coming from equation (62) against timekeeping data, one must also take into account the predicted cosmological contribution to the spin-down of the Earth. If one assumes that the gravitational source of the exterior field (62) is stable with respect to internal collapse (as for a source made of ideal gas), i.e., that possible instabilities generated by the expansion can be neglected, one may model this source as a uniformly expanding sphere. Due to the increase with time of active mass, the angular momenta of test particles moving in the exterior field (62) increase linearly with cosmic scale. This also applies to the angular momentum $L_{\rm s}$ of a spinning source made of ideal gas [9], that is

$$L_{\rm s}(t) = \frac{t}{t_0} L_{\rm s}(t_0), \qquad \frac{dL_{\rm s}}{dt} = \frac{1}{t} L_{\rm s} = (1 + O(2)) H L_{\rm s}, \tag{66}$$

where the term $O(2)$ is of post-Newtonian order and where the locally measured Hubble parameter H is defined by $H \equiv \frac{1}{Nt}$, or equivalently ($\tau_{\mathcal{F}}$ is the proper time of the local FO)

$$H \equiv \frac{t_0}{t} \frac{d}{d\tau_{\mathcal{F}}} (\frac{t}{t_0}) = \frac{ct_0}{t} \Big(\sqrt{B(r)}\Big)^{-1} \frac{d}{dx^0} (\frac{t}{t_0}) = \Big(\sqrt{B(r)} t\Big)^{-1}. \tag{67}$$

Since the moment of inertia $I \propto M R_{\rm s}^2$, where M is the passive mass and $R_{\rm s}$ is the measured radius of the sphere, we must have (neglecting terms of post-Newtonian order)

$$\frac{dR_{\rm s}}{dt} = HR_{\rm s}, \qquad \frac{d\omega_{\rm s}}{dt} = -H\omega_{\rm s}, \qquad \frac{dT_{\rm s}}{dt} = HT_{\rm s}, \tag{68}$$

where $\omega_{\rm s}$ is the spin circle frequency and $T_{\rm s}$ is the spin period of the sphere. (To show equation (68), use the definition $L_{\rm s} = I\omega_{\rm s}$.) This means that the spin period of a sphere made of ideal gas increases linearly with t due to the cosmic expansion. Does this apply to the Earth as well? The Earth is not made of ideal gas, so the cosmic expansion may induce instabilities, affecting its (sidereal) spin period $T_{\rm sE}$. However, here we assume that the Earth's mantle is made of a material which may be approximately modelled as a perfect fluid obeying an equation of state close to linear. Then, if this assumption holds, the Earth should be expanding close to the Hubble rate according to the discussion following equation (49). Moreover, averaged over long time spans, shorter timescale effects of instabilities on $T_{\rm sE}$ should be negligible to a good approximation. We may also assume that there is no significant tidal friction since given the cosmic contribution, this would be inconsistent with the observed so-called mean acceleration $\dot{n}_{\rm m}$ of the Moon (see below). We then get

$$\frac{dT_{\rm sE}}{dt} = HT_{\rm sE}, \qquad \Rightarrow \qquad T_{\rm sE}(t) = \frac{t}{t_0}T_{\rm sE}(t_0). \tag{69}$$

From equation (69) we may estimate a cosmic spin-down of the Earth at the present epoch to be about 0.68 ms/cy (using $H{\sim}2.5{\times}10^{-18}$ s^{-1}). To see if this is consistent with the assumption that the dominant contribution is due to cosmic effects, we may compare to results obtained from historical observations of eclipses and occultations from AD 1000 and onwards. These observations can be used to infer a lengthening of the day of about 1.4 ms/cy [10], whereas an average over the last 2700 years shows a value of about 1.70 ms/cy [11]. But the interpretation of the historical data depends on an assumed vale of $-26''/\text{cy}^2$ for the tidal contribution $\dot{n}_{\rm tid}$ to the mean acceleration $\dot{n}_{\rm m}$ of the Moon (moreover, other significant contributions to $\dot{n}_{\rm m}$ are neglected without justification, see below). This value of $\dot{n}_{\rm tid}$ corresponds to a *calculated* lengthening of the day (using standard theory) of about 2.3 ms/cy [11]; thus the agreement with the values inferred from the historical data is not very good without invoking a secular shortening of the length of the day of non-tidal origin. On the other hand, the QMF yields a value of about $-13.6''/\text{cy}^2$ for $\dot{n}_{\rm m}$ (see below). Reinterpreting the historical data using this value yields a correction to the lenghtening of the day of about -0.62 ms/cy, i.e., the observations could indicate a lengthening of the day of 0.78 ms/cy and 1.08 ms/cy, respectively, rather than the values given above. This means that the values obtained from the historical observations are theory dependent and that the secular spin-down of the Earth may be only about half of the currently accepted value. Note that such a theory dependence also affects the comparison of equation (69) to results obtained from sedimentary tidal rhythmities [12].

Another quantity that can be calculated from equation (69) is the number of days $N_{\rm y}$ in one (sidereal) year T_E. This is found to be constant since

$$T_{\rm E} = N_{\rm y} T_{\rm sE} = \frac{t}{t_0} T_{\rm E}(t_0), \qquad \Rightarrow \qquad \frac{dN_{\rm y}}{dt} = 0. \tag{70}$$

Moreover, the number of the days $N_{\rm m}$ in one (sidereal) month $T_{\rm m}$ can be calculated similarly. We then get

$$T_{\rm m} = N_{\rm m} T_{\rm sE} = \frac{t}{t_0} T_{\rm m}(t_0), \qquad \Rightarrow \qquad \frac{dN_{\rm m}}{dt} = 0. \tag{71}$$

That $N_{\rm y}$ and $N_{\rm m}$ are predicted to be constant is not in agreement with standard (theory dependent) interpretations of paleo-geological data [6, 12]. (The predicted constancy of the ratio $N_{\rm y}/N_{\rm m}$ agrees well with a standard interpretation of the data, though.) In addition to the assumption that active masses do not vary with time, the assumption that $T_{\rm E}$ is constant is routinely used in the interpretation of tidal rhythmities and fossil coral growth data; in particular this applies to [12], where one has explicitly used this assumption when calculating $N_{\rm y}$ from the data. However, values determined directly from the rhythmite record presented in [12] and the predictions given here usually agree within two standard deviations when the predicted variable length of the year is taken into account.

As mentioned above, the mean acceleration of the Moon, $\dot{n}_{\rm m} \equiv \frac{d}{dt} n_{\rm m}$ (where $n_{\rm m}$ is the mean geocentric angular velocity of the Moon as observed from its motion) is a very important quantity for calculating the evolution of the Earth-Moon system. From equation (71) we find the quasi-metric prediction

$$n_{\rm m}(t) = \frac{d\phi_{\rm m}}{dt} = \frac{t_0}{t} n_{\rm m}(t_0), \qquad \Rightarrow \qquad \dot{n}_{\rm m} \equiv \frac{d}{dt} n_{\rm m} = -H n_{\rm m}, \tag{72}$$

and inserting the observed value $0.549''/{\rm s}$ for $n_{\rm m}$ at the present epoch, we get the corresponding cosmological contribution to $\dot{n}_{\rm m}$, namely about $-13.6''/{\rm cy}^2$. This value may be compared to the value $-13.74''/{\rm cy}^2$ obtained from fitting LLR data to a model based on the lunar theory ELP [13]. Note that this second value is the *total* mean acceleration, wherein other modelled (positive) contributions are included. These other contributions amount to about $12.12''/{\rm cy}^2$ and are mainly attributed to the secular variation of the solar eccentricity due to (indirect) planetary perturbations [13]. When these contributions are removed, one deduces a *tidal* contribution $\dot{n}_{\rm tid}$ of about $-25.86''/{\rm cy}^2$ to $\dot{n}_{\rm m}$ [13]. Similar values for $\dot{n}_{\rm tid}$ as inferred from LLR data have been found in, e.g., [14] ($-25.9''/{\rm cy}^2$). We see that in absolute values, the quasi-metric result is smaller than the tidal term inferred from LLR data using standard theory. But since the non-tidal secular contributions to

$\dot{n}_{\rm m}$ are calculated from the ELP theory [13] and not calculated within the quasi-metric framework, and as long as no independent measurements exist confirming these contributions, they can be treated as model-dependent. Thus it is in principle possible to omit both tidal and the traditional non-tidal secular contributions to $\dot{n}_{\rm m}$ and construct a quasi-metric model containing only the cosmic contribution. And as shown above, such a model fits the data well.

We may also use Hubble's law directly to calculate the secular recession $\dot{a}_{\rm qmr}$ of the Moon due to the global cosmic expansion; this yields about 3.0 cm/yr whereas the value $\dot{a}_{\rm tid}$ inferred from LLR using standard theory is (3.82±0.07) cm/yr [14]. To see if the difference between these results can be easily explained in terms of model-dependence, we note that in standard theory, $\dot{n}_{\rm tid}$ represents the value $\dot{n}_{\rm m}$ would have had if the Earth-Moon system were isolated. Therefore $\dot{n}_{\rm tid}$ enters into an expression found by taking the time derivative of Kepler's third law. On the other hand, the quasi-metric model includes the cosmic contribution $\dot{n}_{\rm qmr}$ only, so that quantity enters into a similar expression. Taking into account the fact that active masses increase linearly with time according to quasi-metric theory, we find the relationship

$$\dot{a}_{\rm tid} = \frac{2}{3}\frac{\dot{n}_{\rm tid}}{\dot{n}_{\rm qmr}}\dot{a}_{\rm qmr}, \qquad (73)$$

which is quite consistent with the numerical values given above. It thus seems that there is a simple explanation of the fact that $\dot{a}_{\rm tid}$ as inferred from LLR data using standard theory differs from $\dot{a}_{\rm qmr}$ as found from Hubble's law. In other words, analyzing the LLR data within the QMF yields, to within one standard deviation, that the recession of the Moon follows Hubble's law.

Note once more that whereas the secular recession of the Moon and its mean acceleration have traditional explanations based on tidal friction, these explanations are not confirmed by direct evidence. That is, tidal friction is of nature a mesoscopic phenomenon and it should in principle be possible to measure the tidal energy dissipated in the Earth's oceans. But since no mesoscopic measurements confirming the tidal friction scenario exist so far [15], there are no restrictions on interpreting the secular evolution of the Earth-Moon system as due to cosmological effects.

The apparent constancy of the sidereal year (as indicated by astronomical observations of the Sun and Mercury since about AD 1680) represents the observational basis for adopting the notion that ephemeris time (i.e., the time scale obtained from the observed motion of the Sun) is equal to atomic time (plus a conventional constant), but different from so-called universal time (any time scale based on the rotation of the Earth). However, from equation (65) we see that ephemeris time should be scaled with a factor $\frac{t}{t_0}$

compared to atomic time according to quasi-metric relativity. Moreover, from equation (69) we see that averaged over long time spans, universal time should also be scaled with a factor $\frac{t}{t_0}$ compared to atomic time, as should any conventional constant difference between ephemeris time and universal time. Thus the predicted effect of the spin-down of the Earth and the expansion of the Earth's orbit is a seemingly inconsistency between gravitationally measured time and time measured by an atomic clock. But within the Newtonian framework, any secular changes in the Earth-Moon system are explained in terms of tidal friction (and external perturbations), so seemingly secular inconsistencies between different time scales may be blamed on the variable rotation of the Earth. In practice this means introducing leap seconds. Given the fact that leap seconds are routinely used to adjust the length of the year, the predicted differences between gravitational time and atomic time should be consistent with observations. In particular, the extra time corresponding to an increasing year as predicted from the quasi-metric model may easily be hidden into the declining number of days in a year as predicted from standard theory.

As mentioned earlier, the predicted expansion of the Earth's orbit implies that the length of the year increases with about 2.5 ms at the present epoch. This corresponds to a heliocentric mean angular acceleration $\dot{n}_{\rm E}$ of about $-1.0''/{\rm cy}^2$. But this is inconsistent with the calculated orbital motion of the Earth-Moon barycenter from the ELP theory, which yields $\dot{n}_{\rm E}$ of about $-0.040''/{\rm cy}^2$ [13]. To see if this discrepancy can be explained by model-dependent assumptions inherent in the ELP theory, we may estimate the expansion $\dot{a}_{\rm E}$ of the Earth's orbit radius corresponding to the ELP value of $\dot{n}_{\rm E}$ and then compare this estimate to an independent result. We do the estimate by comparing the ELP and the quasi-metric models using an equation similar to equation (73). We also assume that external perturbations of the Earth's orbit can be neglected in the quasi-metric model. From Kepler's third law we find an equation similar to equation (73) relating the expansion of the Earth's orbit radius $\dot{a}_{\rm E}$ and $\dot{n}_{\rm E}$ as calculated from ELP theory to their counterparts as calculated from quasi-metric theory. Using Hubble's law to calculate $\dot{a}_{\rm E}$ we find a value of about 1.2×10^3 m/cy from quasi-metric theory. Then, using the values for $\dot{n}_{\rm E}$ mentioned above we estimate $\dot{a}_{\rm E}$ as calculated from ELP theory to be about 32 m/cy from the relationship between said quantities. Interestingly, an analysis of all available radiometric measurements of distances between the Earth and the major planets yields a value of 15$\pm$4 m/cy for $\dot{a}_{\rm E}$ [16], i.e., about half of the value calculated above. So, contrary to what is asserted in [16], an explanation of this effect based on the cosmic expansion is not at all shown to be inadequate since most, if not all, of the substantial difference between $\dot{a}_{\rm E}$ as calculated from Hubble's law on the one hand and that inferred

from radiometric data on the other hand, could be due to gross modelling errors. Further evidence for the existence of modelling errors due to local cosmic expansion comes from optical observations of the Sun, indicating an inconsistency in modern ephemerides which may be interpreted as an error of about $1''$/cy in n_{E} [17, 18].

In this section we have seen that the predicted effects of the cosmic expansion on the Earth-Moon and solar system gravitational fields have a number of observable consequences, none of which is shown to be in conflict with observations so far, even though superficially, it would seem that some are. That is, in every case where there is an apparent conflict between quasi-metric predictions and observations, the discrepancies can be explained in terms of model-dependent assumptions made when analyzing the data. In the next section we will see that a similar situation exists for the predicted versus the observationally inferred time variation of the gravitational "constant".

4.3 The secular increase of active mass

In quasi-metric relativity active mass varies throughout space-time (but *not* in the Newtonian limit of the QMF, since this variation is defined in terms of a varying scale factor). In particular there is a secular increase as seen from equations (8) and (48). This is equivalent to a secular increase of the gravitational "constant" G_{C} as measured in a local gravitational test experiment (e.g., a Cavendish experiment). From equation (8) we get the predicted time variation

$$\frac{G_{\mathrm{C},t}}{G_{\mathrm{C}}} = \frac{1}{t} = (1+O(2))H \approx 8\times 10^{-11}\,\mathrm{yr}^{-1}, \qquad (74)$$

for the present epoch. However, laboratory gravitational experiments are nowhere near the experimental accuracy needed to test this prediction. On the other hand, space experiments in the solar system (e.g., ranging measurements) and observational constraints on solar models from helioseismology are claimed to rule out any possible fractional time variation of G larger than about 10^{-12} yr^{-1}. See e.g. [19-21] and references therein. It thus may appear as the prediction (74) is in conflict with experiment. But as we shall see in the following, this does not follow.

To illustrate the difference between metric and quasi-metric theory when it comes to the effects of a varying G on the equations of motion, we note that in the weak field limit of metric theory we may set $G(t) = G(t_0) + \dot{G}(t_0)(t-t_0) + \cdots$ directly into the Newtonian equation of motion. For an inertial test particle this yields (using a Cartesian coordinate

system)

$$\frac{d^2x^j}{dt^2} = U\Big[t, x^k, G(t_0)(1 + \frac{\dot{G}(t_0)}{G(t_0)}(t - t_0) + \cdots)\Big]_{,j}\,, \tag{75}$$

leading to an extra, time-dependent term in the coordinate acceleration of objects. It is the presence of such an extra term which is ruled out to a high degree of accuracy according to the space experiments testing the temporal variation of G. That is, one tests a combination of the predicted changes of the solar system scale and orbit periods T which are predicted to vary as $\frac{\dot{T}}{T} = -2\frac{\dot{G}}{G}$. This follows from Kepler's third law since one requires that the conservation of angular momentum takes the form $\dot{n} = -2\frac{\dot{a}}{a}n$ for any object with mean heliocentric angular velocity n, mean angular acceleration $\dot{n}$ and fractional change of orbit radius $\frac{\dot{a}}{a}$. But this requirement is inconsistent with quasi-metric gravity since we see from equation (31) that the conserved quantity is given by $\frac{t_0}{t}\ell^2 n$ (where $\ell \equiv \frac{t}{t_0}r$ and where corrections of post-Newtonian order have been neglected), and not by $r^2 n$. This yields $\dot{n} = -\frac{\dot{a}}{a}n$ and thus equation (74) when applying Kepler's third law.

Contrary to metric theory, no such extra term as shown in equation (75) is present in the weak field limit of quasi-metric theory since U does not depend on t. An example of this can be seen from equations (59) and (62), where $U \approx \frac{M_{t_0}G}{r} = \frac{M_t G}{\ell}$ does not depend on t. (On the other hand U may depend on x^0, but any variation of $M_t G$ with x^0 is not (directly) due to cosmology.) However, from equations (65) and (74) we see that in quasi-metric theory we have $\frac{\dot{T}}{T} = \frac{\dot{G}_{\mathrm{C}}}{G_{\mathrm{C}}}$. But as we have seen in section 4.2, in combination with the predicted scale changes due to the cosmic expansion this is not inconsistent with observations.

In the weak field limit of metric theory one may calculate the effects on stellar structure coming from a possible variation of G. Such effects are found by putting a variable G directly into the Poisson equation. That is, a change in G directly induces a change in the Newtonian potential yielding a change in star luminosity. Such calculated changes in luminosity are tightly constrained from their effects on star models, which can be compared to observations, e.g., data obtained from helioseismology. On the other hand, in quasi-metric gravity the effect of the secular increase of active mass cannot be separated from the cosmic expansion, so their total effect is to *decrease* the density (of a body made of ideal gas) with cosmic epoch but such that the Newtonian potential is unchanged. (To see how this works, recall how equation (12) reduces to equation (43) and take its Newtonian limit. Discover that the resulting Poisson equation is unaffected by the combination of expansion and increasing G.) Thus for a main sequence star there will be

approximately no change in luminosity except for that due to the increase of scale (i.e., the increase in luminosity due to the increasing surface area of the expanding star).

We conclude that all space experimental tests of the secular variation of G are based on the assumption that this variation is present explicitly in the Newtonian potential. (This also holds for tests based on stellar structure and in particular restrictions coming from helioseismology.) However, this assumption (and in particular equation (75)) does not hold in quasi-metric theory. Hence, the interpretations of these tests are explicitly theory dependent and the prediction made in equation (74) has not been shown to be in conflict with current experimental results, despite the variety of tests apparently showing otherwise. Finally, notice that any cosmological constraints on the secular variation of G found within the metric framework are utterly irrelevant for quasi-metric theory.

5 Conclusion

In this paper we have shown that according to the QMF, gravitationally bound, metrically static systems are predicted to expand with the Universe. Interior to sources the metrically static condition applies whenever the equation of state is of the form $p \propto \rho_{\rm m}$ (fulfilled, e.g., for an ideal gas). When it is not the global cosmic expansion is predicted to induce instabilities violating hydrostatic equilibrium. For any such source mass, currents are set up to compensate and the system cannot be metrically static. (An example of this is a body made of degenerate star matter, e.g. a white dwarf, which is predicted to *shrink* with epoch.)

According to the QMF, the predicted effects on gravitationally bound systems of the global cosmic expansion have a number of observable consequences, none of which has been shown to be in conflict with observations. That is, it seems that at this time no model independent evidence exists that may rule out the possibility that the size of the solar system (measured in atomic units) expands according to the Hubble law; on the contrary the quasi-metric model fits some observational data more naturally than traditional models do. (But note that predictions coming from the QMF fit these data naturally only as long as the data are interpreted in a manner consistent with the QMF.)

Some examples of observations being naturally explained within the non-metric sector of the QMF have been discussed in this paper; e.g., the spin-down of the Earth [10, 11], the recession of the Moon and its mean acceleration [13, 14], and the newly discovered secular increase of the astronomical unit [16]. Also the so-called "Pioneer effect" has a natural explanation within the QMF [22]. Thus the non-metric sector of the QMF has considerable predictive power in the solar system, since it makes it possible to explain

from first principles a number of seemingly unrelated phenomena as different aspects of the same model. On the other hand, explanations of these phenomena coming from standard theory are invariably *ad hoc*; such explanations always involve free parameters and mechanisms invented to explain each phenomenon separately. Such an approach is untenable according to Occam's razor.

So fact is that several observations in the solar system represent evidence that space-time is quasi-metric. Moreover, metric gravity (and General Relativity in particular) fails to address the challenge represented by these observations. And the main reason that this challenge has not been recognized as important, is that experimental gravity in the solar system is analyzed within a weak field formalism (the so-called PPN-formalism) where it is inherently assumed that space-time must be modelled as a Lorentzian manifold, and consequently that the cosmic expansion will be unmeasurably small at the scale of the solar system. This situation may change in the future, when solar system gravitational experiments reach a precision level where the quasi-metric effects can no longer reasonably be "explained" by adding *ad hoc* hypotheses to metric theory.

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